Uniform Waveguides with Arbitrary Cross-Section Considered by the Point-Matching Method

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Abstract—**The point-matching method applies to the problem of wave propagation in many uniform waveguides of very general cross-sections. The boundary conditions are satisfied at a finite number of points on the guide wall only. This method applies when the contour of the cross section of the guide is a closed curve, the function of which is single-valued. The validity of the point-matching method is demonstrated qualitatively. Examples show that accurate values of cutoff wave numbers can be achieved easily.**

I. Introduction

THE PROBLEM of electromagnetic wave propagation in hollow-piped waveguides has been of considerable interest in recent years. While the basic theory goes back to the days of Maxwell, the applications to certain types of waveguides have been accomplished in the last few decades.

Guides of certain shapes, rectangular, circular, elliptical, and parabolic have been studied extensively. The solutions of the wave equations subjected to the pertinent boundary conditions for such configurations are relatively easy to obtain because of the separability of the equations in the cross-sectional coordinate systems. Since the previously mentioned waveguides cannot fulfill many modern engineering requirements, knowledge of waveguides with complicated cross sections is desirable. Unfortunately, the method of separation of variables fails for problems other than the conventional types. Consequently, approximate techniques must be utilized.

Waveguides with somewhat complicated cross sections were first investigated by Cohn [1] in a study of ridge guides, in 1947. After that, many authors [2]–[6] studied the properties of waveguides with different cross sections. Recently, Meinke, et al. [7], and Tischer and Yee [8], [9] used the conformal mapping method along with various approximation techniques to solve the boundary-value problems for guides with general cross sections. However, analytical conformal transformations, in general, cannot be easily found.

In this paper an approximate technique, called the point-matching method, is introduced which can be applied to uniform waveguides of very general cross sections. The only limitation on the cross section is that the representative function of the closed contour is single-valued in the radical direction.

The calculations of the properties of waveguides by this method are simple if a digital computer is accessible. The accuracy of the method is demonstrated by applying it to a square guide. An error of 0.05 percent for the cutoff wave number of the dominant mode can easily be obtained. The degeneracies can also be located. With the knowledge of the field expressions, the power transmitted, and the attenuation constant due to the finite conductivity of the guide wall may be evaluated by numerical methods.

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II. Theory

Consider the air-filled hollow-piped uniform waveguide of an arbitrary cross section with a coordinate system as shown in Fig. 1(a). Let an electromagnetic wave propagate in the z direction. The wave equation for this system, assuming a time-harmonic dependence [exp $(j\omega t)$], is given by [10]

$$(\nabla_t^2 + k^2)\psi = 0 \tag{1}$$

where

$$k^2 = k_0^2 - k_z^2$$
$$k_0^2 = \omega^2\mu_0\epsilon_0$$
$$k_z = 2\pi/\lambda_g.$$

The quantity λ_g is the guide wavelength, and ∇_t^2 is the two-dimensional transverse Laplacian operator. The wave function $\psi = H_z$ for TE (transverse electric) wave modes, and $\psi = E_z$ for TM (transverse magnetic) wave modes. The eigenfunction ψ must satisfy either Dirichlet or Neumann boundary conditions. The eigenvalue problem consists of finding suitable solutions of (1) subjected to the pertinent boundary conditions, and determining the eigenvalues k. With the knowledge of longitudinal components E_z or H_z, the transverse field components can be computed by [10]

$$\overline{E}_t = (jk_z/k^2)[-\nabla_t E_z + (\omega\mu_0/k_z)\bar{z} \times (\nabla_t H_z)] \tag{2}$$

$$\overline{H}_t = (-jk_z/k^2)[(\omega\epsilon_0/k_z)\bar{z} \times (\nabla_t E_z) + \nabla_t H_z] \tag{3}$$

where $\bar{z}$ is a unit vector in the z direction. The guide impedance, power transfer, and the attenuation due to the finite conductivity of the guide walls can be evaluated by conventional methods.

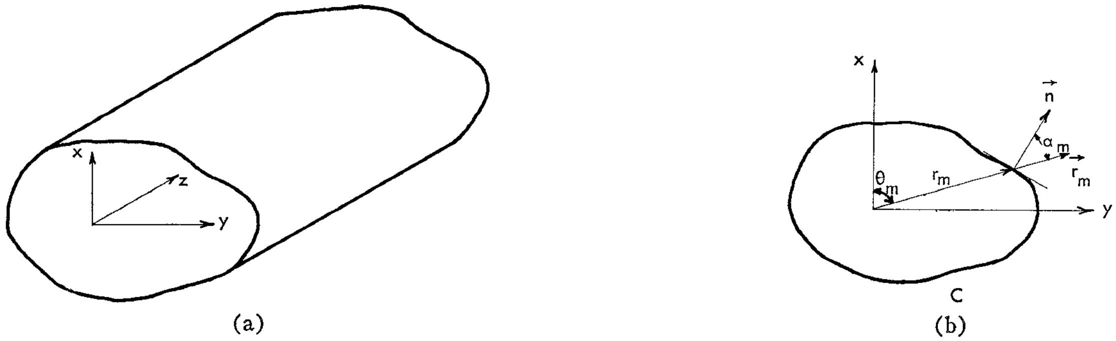

Fig. 1. (a) A waveguide with an arbitrary cross section and a relevant coordinate system. (b) The angle α_m at point (r_m, θ_m) on the cross-sectional contour.

For waveguides with simple geometrical cross sections, the eigenfunction ψ is obtained by solving (1), in which the variables are usually separable. If the guide's cross-sectional coordinate system is not one of the separable coordinate systems, namely, rectangular, polar, elliptical, or parabolic, the method of separation of variables fails. However, the general solution of (1) in one of the separable coordinate systems is still a solution of a waveguide with an arbitrary cross section. The problem is then how to impose the boundary conditions for this solution and how to find the eigenvalues.

It is convenient to express the general solution of (1) in polar coordinates as follows:

$$\psi = \sum_{n=0}^{\infty} J_n(kr)(A_n \cos n\theta + B_n \sin n\theta) \tag{4}$$

where r and θ are the polar coordinates. J_n is the Bessel function of first kind, A_n and B_n are constants to be determined by the boundary conditions. Assuming that the series in (4) converges rapidly and uniformly for the cases under consideration, the solution may be approximated by a finite number of terms

$$\psi \approx \sum_{n=0}^{N} J_n(kr)(A_n \cos n\theta + B_n \sin n\theta). \tag{5}$$

In general, it is difficult to have (4) or (5) satisfy the boundary conditions at every point around the closed contour C of a generally shaped guide such as the one sketched in Fig. 1(a). However, it is possible to require (5) to satisfy the boundary conditions at a finite number of points, namely $2N+1$. Let the points (r_0, θ_0), (r_1, θ_1), (r_2, θ_2), $\cdots$, (r_m, θ_m), $\cdots$, (r_{2N}, θ_{2N}) be a set of chosen points around the cross section. The boundary conditions at these points for TM modes require

$$\sum_{n=0}^{N} (A_n \cos n\theta_m + B_n \sin n\theta_m) J_n(kr_m) = 0 \tag{6}$$

and for TE modes require

$$\bar{n} \cdot \nabla_t \sum_{n=0}^{N} (A_n \cos n\theta_m + B_n \sin n\theta_m) J_n(kr_m) = 0 \tag{7}$$

where $m=0, 1, 2, \cdots, 2N$, and $\bar{n}$ is the unit vector normal to the surface. More precisely (7) may be written in the following form:

$$kr_m \sum_{n=0}^{N} (A_n \cos n\theta_m + B_n \sin n\theta_m) J_n'(kr_m)$$

$$+ \tan \alpha_m \sum_{n=0}^{N} n(-A_n \sin n\theta_m + B_n \cos n\theta_m) J_n(kr_m) = 0 \tag{8}$$

where $\cos \alpha_m = \bar{n} \cdot \bar{r}_m$, and $\bar{r}_m$ is the unit vector in the r direction at point (r_m, θ_m) as shown in Fig. 1(b). The angle α_m may be described in the following manner. Let the contour C of the cross section be described by

$$f(x, y) = 0;$$

where

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

then

$$\tan \alpha_m = -(1 + F_m \tan \theta)/(F_m - \tan \theta)$$

where

$$F_m = F(x_m, y_m) = \left.\frac{dy}{dx}\right|_{y=y_m;\ x=x_m}.$$

Each of (6) and (8) forms a system of $2N+1$ homogeneous algebraic equations of $2N+2$ unknowns; namely, A_n, B_n, and k. To obtain nontrivial solutions of the expansion coefficients A_n and B_n, the determinant of these coefficients must be zero. That is

$$D(k) = \det |d_{ij}| = 0 \tag{9}$$

where

$$d_{ij} = J_i(kr_j) \cos i\theta_j; \qquad i = 0, 1, 2, \cdots, N$$

$$d_{ij} = J_{i-N}(kr_j) \sin (i-N)\theta_j; \qquad i = N+1, \cdots, 2N$$

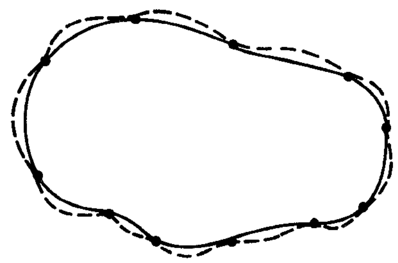

Fig. 2. Contours of the waveguide's cross section; ——— original; — — — — — — obtained by (10).

for TM modes; and

$$d_{ij} = kr_j \cos i\theta_j J_i'(kr_j) - i \tan \alpha_j \sin i\theta_j J_i(kr_j); \quad i = 0, 1, 2, \cdots, N$$

$$d_{ij} = kr_j \sin (i - N)\theta_j J'_{i-N}(kr_j) + (i - N) \tan \alpha_j \cos (i - N)\theta_j J_{i-N}(kr_j); \quad i = N + 1, \cdots, 2N$$

for TE modes. The roots of (9) determine the eigenvalues k. There are an infinite number of roots, each of which corresponds to a wave mode. In general, the roots of higher order modes (larger values of k) calculated by this numerical method are less accurate than the roots of lower order modes. The accuracy may be improved, however, by extending N to larger numbers. In other words, more points on the contour are chosen to satisfy the boundary conditions.

Having determined the eigenvalue k for a specific mode, the expansion coefficients A_n and B_n can readily be found from (6) or (8).

III. Qualitative Discussions of the Point-Matching Method

In the previous discussion, the eigenfunction ψ is expressed in terms of a set of functions

$$\phi_n \left[= J_n(kr) \begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix} \right],$$

and the boundary conditions are satisfied at only a finite number of points around the conducting surface. The completeness of the set of functions $\{\phi_n\}$ and the fulfillment of boundary conditions are not established, and will not be proved in this paper. Instead, the validity of the point-matching method will be presented from a qualitative point of view as follows.

For a given TM wave mode, the expansion coefficients A_n and B_n, and the eigenvalue k in (5) can be found by the method described previously. Since (5) is a real analytic function of r and θ, and is periodic in the angular direction, then the equation

$$\sum_{n=0}^{N} (A_n \cos n\theta_c + B_n \sin n\theta_c) J_n(kr_c) = 0 \tag{10}$$

describes a closed curve. The subscript c denotes the boundary. The curve connects the chosen points, and may deviate from the original contour of the cross section as shown in Fig. 2. Having described the field by (5), and the boundary conditions by (10), then (5) and the value of k are the exact eigenfunction and the eigenvalue respectively, for a waveguide of this TM mode with a cross section defined by (10). If the intervals between the chosen points are sufficiently small, the deviation between the original curve and that described by (10) is expected to be negligibly small. Therefore, (5) and k computed by the point-matching method can be considered as a good approximate solution for the original waveguide.

Similar argument can be applied when (5) represents H_z for the TE wave modes. When the boundary conditions are applied at the chosen points, the resulting equation is

$$kr_c \sum_{n=0}^{N} (A_n \cos n\theta_c + B_n \sin n\theta_c) J_n'(kr_c) + \tan \alpha \sum_{n=0}^{N} n(-A_n \sin n\theta_c + B_n \cos n\theta_c) J_n(kr_c) = 0 \tag{11}$$

where

$$\tan \alpha = -\left(1 + \tan\theta \frac{dy}{dx}\right) \Big/ \left(\frac{dy}{dx} - \tan\theta\right)\Bigg|_C = \frac{1}{r}\frac{dr}{d\theta}\Bigg|_C.$$

The solution of (11), which is a nonlinear first-order differential equation, is not known at the present time. Since the equation is analytic everywhere, it can be linearized so that it is valid only in small regions around the chosen points. If the intervals between the points are made sufficiently small, the neighboring regions overlap as illustrated in Fig. 3. Solutions start at the chosen points along the direction perpendicular to the known normal vectors at the points. Two neighboring normal vectors are assumed almost to be parallel. Since the trajectories of a nonlinear differential equation cannot intersect each other for a normal system [11], the solu-

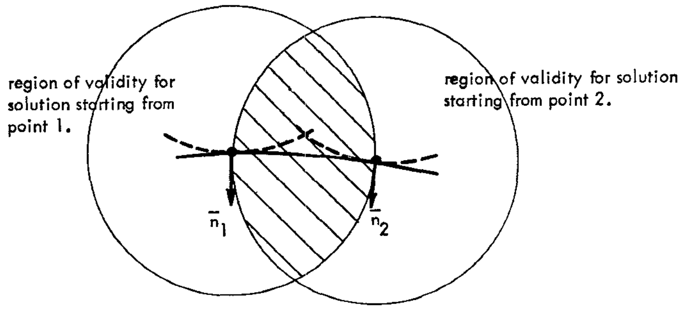

Fig. 3. Solutions of (11) at two adjacent points. ——— exact; ——— - ——— ——— linearized.

tions extended from two neighboring points are then connected smoothly. The possibility of intersection is due to the approximation of linearization. This argument is confirmed by the excellent agreement of the cutoff wave numbers calculated by this method with the exact solutions as shown in Section V. Therefore, (11) represents a closed contour at which H_z satisfies Neumann boundary condition. This contour is expected to be a close approximation of the original cross section if the intervals between the chosen points are sufficiently small.

IV. Limitations of the Method

While the point-matching method offers a new way for solving boundary-value problems of waveguides of arbitrary cross sections, it does have limitations to its applications which can be seen from previous analysis, namely:

1) The boundary line of the waveguide's cross section must be closed, since (10) and (11) describe closed contours which are good approximations of the waveguide's cross section.

2) The representative function of the closed curve must be single-valued in the r direction. For every constant θ, there is one and only one value of r on the same contour [(10) and (11) actually describe a family of closed contours; one encloses the other] where (10) and (11) are satisfied.

V. Examples

The first example one may think of is the circular waveguide. If a is the radius, then the radial coordinates of the chosen points around the periphery are $r_0 = r_1 = r_2 = \cdots = r_{2N} = a$, and (9) is readily reducible to the exact solutions.

To demonstrate the accuracy of the point-matching method applied to waveguides with noncircular cross sections, the cutoff wave numbers of elliptical and square waveguide were calculated and compared with the exact solutions.

The cross section of the elliptical waveguide under consideration is defined by

$$(x/1.5431)^2 + (y/1.1752)^2 = a^2.$$

Since the ellipse is symmetric with respect to the x axis, the field distributions inside the guide are either symmetric or antisymmetric. In the following discussion, wave modes with longitudinal field component symmetric with respect to the x axis are called even modes, and the modes with longitudinal field antisymmetric with respect to the x axis are called odd modes. With this classification, the sine terms in (5) are omitted for even modes, while the cosine terms are omitted for odd modes. For the present ellipse, only nine points were chosen on the upper periphery for even modes, and seven points were chosen for odd modes. Table I lists the cutoff wave numbers for the TM and TE modes calculated by the point-matching method and compared with those calculated from Chu's curves [12], and those calculated by the method of conformal mapping [9]. Note that the accuracy of the wave numbers calculated from Chu's curves are good only to two places.

Similar calculations for a square waveguide, as a second example, show that excellent accuracy can be achieved using the point-matching method, even though sharp corners are present. If a square of width $2a$ is placed with its center at the origin of the coordinate system, then symmetry with respect to the x axis is obtained. The cutoff wave numbers ka of TM modes calculated by the point-matching method are listed in Table II and compared with the exact values. Thirteen points were used to calculate the values of cutoff wave numbers for both even and odd modes. The error in the calculation for the lowest order mode TM_{11} is less than 0.05 percent. From Table II, the double degeneracy of TM_{13} and TM_{31} can be observed, when finding the roots of (9). The function $D(k)$ passes through a minimum which is almost zero at $ka = 4.9670$. In the case of TM_{24} and TM_{42}, the two roots are very close together. The separation, however, using eleven points is larger than that when thirteen points are used for the calculation.

The cutoff wave numbers of even TE modes calculated by using thirteen points are listed in Table III. The double degeneracy of TE_{20} and TE_{02} is also observed.

At the corner points of the preceding calculations, the normals were taken in the direction of the bisectors of the angles. This choice is shown to be valid by the excellent agreement between the exact and the approximate values.

TABLE I

Comparison of Cutoff Wave Numbers, *ka* of an Elliptical Waveguide

Mode	Classification	Point-Matching Method	Chu	Conformal Mapping
TM_{01}	even	1 817	1.8	—
TM_{11}	even	2.694	2.7	—
TM_{11}	odd	3.080	3.1	3.08
TM_{21}	odd	3.873	—	3.88
TE_{11}	even	1.204	1.2	1.204
TE_{01}	even	3.023	3.0	—

TABLE II

Comparison of TM Cutoff Wave Number *ka* of a Square Waveguide of Width $2a$

TM	Classification	Point-Matching Method	Exact
11	even	2.2204	2.2214
21	even	3.5102	3.5124
13,31	even	4.9670	4.9673
23	even	5.6505	5.6636
41	even	6.5523	6.4766
33	even	6.6651	6.6643
43	even	7.9779	7.8540
22	odd	4.4428	4.4429
32	odd	5.6622	5.6636
14	odd	6.4763	6.4766
24,42	odd	7.0243 7.0253	7.0248
34	odd	7.8102	7.8540
52	odd	8.4053	8.4590

TABLE III

Comparison of Even TE Wave Numbers *ka* of a Square Waveguide of Width $2a$

TE	Point-Matching Method	Exact
10	1.5708	1.5708
20,02	3.142	3.1416
12	3.5129	3.5124
22	4.4427	4 4429
30	4.7138	4.7124
32	5.6671	5.6636
40	6.4673	6.2832

VI. Conclusion

The point-matching method offers a new way of solving boundary-value problems of hollow-piped uniform waveguides when the method of separation fails. This numerical technique was applied to square and elliptical waveguides in order to demonstrate that excellent accuracy in calculating the cutoff wave numbers can easily be determined. The boundary conditions were imposed at only a finite number of points along the contour of the cross section. The boundary conditions at points other than the chosen ones are satisfied approximately if the contour of the waveguide's cross section is closed and its representative function is single-valued. When a particular waveguide is considered by the point-matching method, then there exists a contour where the boundary conditions are satisfied, and it deviates from the original cross section. The deviation will be improved by increasing the number of chosen points. The degeneracies can also be observed by this method.

Having found the cutoff wave numbers for a waveguide, the wave function can then be determined. The transferred power, attenuation constant, the field pattern, and the current distribution are readily obtainable by numerical methods.

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